SECT. II.

DEFINITIONS.

1. A FLOWING quantity is one that continually increaſes or decreaſes, and in ſuch a manner that ſome time is requiſite to make any increment or decrement.

2. THE Fluxion of a flowing quantity is its rate or ſwiftneſs of increaſe or decreaſe.

3. THE change of a flowing quantity is the difference between the flowing quantity itſelf, and its value at a particular inſtant of time.

4. THE time in which a change is made, is the time the flowing quantity takes to alter from a prior to a ſubſequent value, whoſe difference is the change.

5. THAT change is ſaid to vaniſh at a given inſtant, which is the difference between the flowing quantity before that inſtant and its value then; and that change is ſaid to begin to ariſe at a given inſtant, which is the difference between the flowing quantity after that inſtant, and its value at that inſtant.

ILLUSTRATION.

LET y be a quantity continually increaſing, A its value at a given inſtant; then $A-y$ will repreſent its change [taken from that inſtant] ſo long as y is leſs than A; and this change will vaniſh when y becomes equal to A; and by the continued

continued increase of y, there will then begin to arise another change which is ever afterwards equal to $y - A$.

Def. 6. If there be two permanent quantities A and B, and two other flowing Quantities a and b, and the ratio of a to b be always, during a given time, that of the sum or difference of the first permanent quantity A, and another flowing quantity x to the sum or difference of the second permanent quantity B, and another flowing quantity y, and at the end of the given time all the flowing quantities vanish; then the ratio of the permanent quantities A and B, is the last or ultimate ratio of the vanishing quantities a and b; which I thus express, ult. $a : b :: A : B$. *i. e.* if $a : b :: A \mp x : B \mp y$ always, during the time T, and at the end of that time, a, b, x, y all vanish; then ult. $a : b :: A : B$.

7. If other things being supposed the same, a, b, x, y, be each equal to nothing at the beginning of the time T, then the ratio of A to B is the first ratio of the nascent quantities a and b; which I thus express, p^{mo} $a : b :: A : B$.

Remark. These two definitions are in effect the same with those given by Sir *Isaac Newton*, and can't be disputed; for whether a and b, properly speaking, have any proportion as they arise, or vanish, yet A and B have; and that I am at liberty to call by what name I please.

Coroll. 1. The ultimate ratio of two vanishing quantities, is a determinate ratio; *i. e.* if ult. $a : b :: A : B$, and ult. $a : b :: A : N$. B is $= N$. For since ult. $a : b :: A : B$; therefore, by *Def.* 6. $a : b :: A \mp x : B \mp y$ during a given time T, at the end of which the flowing quantities a, b, x, y, all vanish; and by the same *Defin.*

Defin. $a : b :: A \mp v : N \mp z$, during a given time t, at the end of which the flowing quantities a, b, v, z, all vanish; and the end of t and T coincide, because that is the instant when a and b vanish. Taking therefore the shorter of these times t, during that $A \mp x : B \mp y :: A \mp v : N \mp z$. Now because x, y, z, v, change only as flowing quantities, and vanish together at the end of the time t, before the end of the time t they will be all less than any assignable quantity D [for suppose any of them never less than D before that instant, then a decrement as large as D must be made instantaneously, in order to its vanishing at that instant, which is contrary to the supposition of its changing only as a flowing quantity.] But since x, y, v, z, may be as small as you please, and yet the analogy $A \mp x : B \mp y :: A \mp v : N \mp z$ be preserved, it is plain that $A : B :: A : N$; and that therefore $B = N$. Q. E. D.

Coroll. 2. In like manner it may be proved that the first ratio of two arising quantities is a given ratio.

Axiom I.

The sum and difference of two flowing, and the fourth proportional to three flowing quantities; and therefore a quantity any how made up of given and flowing quantities, must be itself a flowing or permanent quantity; *i. e.* some time is requisite to its receiving any increment or decrement, because a change in it must imply a change at least in some one of them.

Coroll. 3. If $a : b :: A \mp x : B$ during the time T; then if, at the end of the time T, the flowing quantities a, b, x vanish, ult. $a : b :: A : B$; and if, at the beginning of T, they are equal

to

to nothing, p^{mo} $a:b::A:B$. For because $a:b::A \mp x:B$, therefore $a:b::A+a \mp x:B+b$ during the time T, and a and b, and $a \mp x$ are flowing quantities all vanishing together, or all arising together; and therefore, by *Defin.* 6 and 7, in the former case ult. $a:b::A:B$, and in the latter p^{mo} $a:b::A:B$.

Coroll. 4. IF $a:b::A:B$ always, a and b vanishing or arising together, then ult. $a:b::A:B$; or p^{mo} $a:b::A:B$. This is proved as the foregoing.

Coroll. 5. IF the ratio of two quantities continually decreasing approach continually nearer and nearer to a given ratio, and by so doing at length come nearer to it than by any assignable difference, that given ratio is the last ratio of the flowing quantities when they vanish. Let the flowing quantities be a and b, the given ratio $A:B$. Then supposing $a:b::A:B \mp x$ always, as a and b decrease, x will also continually decrease in the same manner as a flowing quantity, by *Ax.* 1. and x may be as small as you please, because the ratio of a to b may approach nearer to that of A to B, than by any assignable difference. And therefore when a and b vanish, x must vanish with them; [for if it then be equal to D, it never before was less than D, since it was continually decreasing; whereas before it was smaller than any assignable quantity] and therefore, by *Coroll.* 3. ult. $a:b::A:B$.

Coroll. 6. IF two flowing quantities a and b arise from nothing at a given instant, and the nearer the time be taken to that of their rise, so much the nearer is their ratio to that of A to B; and the time may be taken so small, that the ratio

ratio of a to b shall differ from that of A to B, less than by any assignable difference, in this case, p^mo^ $a : b :: A : B$. The Proof of this is similar to that of the foregoing.

Remark. IN these two last corollaries you have Sir *Isaac Newton*'s description of the ultimate ratio of vanishing quantities, and the * prime ratio of nascent or arising ones, which is in effect the same with that given in *Defin.* 6, 7. But I chose to make this little alteration for the greater expedition in practice. And it is to be noted, that tho' Sir *Isaac*, to explain what he means by the ultimate ratio of vanishing quantities, describes it as in *Coroll.* 4. yet in practice he seems evidently enough to make use of one similar to that of *Defin.* 6.

Coroll. 7. IF ult. $a : b :: A : B$, then ult. $a + b : b :: A + B : B$. For since ult. $a : b :: A : B$, by *Defin.* 6. a time T may be assumed; during which $a : b :: A \mp x : B \mp y$; and at the end of which

* We have not, as I know of, any direct definition of these prime ratio's, but we are left to form it from that most accurate definition of the ultimate ratio's of vanishing quantities; which we have at the latter end of *Schol. Lemm.* II. *Princ.* and which is so plain, that I wonder how our author could help understanding it; which had he done, I am apt to think that all his *Analyst* says concerning the proportion of quantities vanishing with the quantities themselves, had never been heard: For according to this definition, we are not obliged to consider the last ratio as ever subsisting between the vanishing quantities themselves. But between other quantities it may subsist, not only after the vanishing quantities are quite destroyed, but before when they are as large as you please. And the reason we consider quantities as decreasing continually till they vanish, is not in order to make, but to find out, this last ratio. Sir *Isaac Newton* does indeed say that this last ratio is the ratio with which the quantities themselves vanish; but whether he herein speaks with the utmost propriety or not, is a mere nicety on which nothing at all depends.

which the flowing quantities a, b, x, y, all vanish; and therefore also during that time, $a + b : b :: A + B \mp x \mp y : B \mp y$; and at the end of that time also, the four quantities $a + b$, b, $x \mp y$, and y, which are flowing quantities, by *Axiom* 1. vanish: wherefore, by *Defin.* 6. ult. $a + b : b :: A + B : B$.

Coroll. 8. IF ult. $a : b :: A : B$, then ult. $a - b : b :: A - B : B$; A being greater than B.

Coroll. 9. IF ult. $a : b :: A : B$, and ult. $b : d :: B : D$; then also by equality, ult. $a : d :: A : D$.

Remark. THESE two last corollaries are demonstrated by the like contrivance as the sixth; and the like propositions are true, and in like manner to be proved concerning prime ratio's.

Axiom 2. A QUANTITY flows uniformly, or with a permanent Fluxion, when the changes made in it are always proportional to the times in which they are made.

Axiom 3. THE Fluxions of quantities uniformly flowing, are always in the proportion of their synchronal changes; and therefore, because this is a given ratio (by *Coroll.* 4. *Defin.* 5 and 6.) in their ultimate ratio when they vanish, or their prime ratio when they arise.

Axiom 4. IF thro' any time two quantities be generated, or changes be made in two quantities, one with an uniform Fluxion or rate of increase or decrease, and the other with a Fluxion continually increasing; then the ratio of the quantity generated, or the change made by the uniform or permanent Fluxion to the quantity generated, or the change made by the continually increasing Fluxion, will be always less than the ratio of the Fluxion of the former is to the Fluxion of the latter at the beginning of the time, and greater than the ratio which the Fluxions are in to one another at the end of the time.

Axiom 5. BUT if, other things being ſuppoſed the ſame, the latter Fluxion continually decreaſe, the ratio of the quantity generated, or the change made by the uniform or permanent Fluxion to the quantity generated, or the change made by the continually decreaſing Fluxion, will be always greater than the ratio of the Fluxion of the former is to the Fluxion of the latter at the beginning of the time, and leſs than the ratio of the Fluxion of the former to the Fluxion of the latter at the end of the time. Both theſe axioms are plain conſequences from this obvious truth, that if any thing increaſe with an accelerated velocity or ſwiftneſs, the increaſe made in a given time will be greater than if it had all along increaſed with the ſame ſwiftneſs it did at the beginning, and leſs than if it had all along increaſed with the ſwiftneſs it did at the end of it; and that the contrary will happen if the velocity be continually retarded. And the reader will pleaſe to take notice, that as the foregoing axioms are the ſole principles on which the doctrine of Fluxions is built, ſo they were generally allowed, before this doctrine was ever thought of, in relation to the velocities of moving bodies, and the ſpaces deſcribed by their motion. The only thing new in this caſe is, that the idea of velocity is render'd more general, and, in order to prevent ambiguity, the word Fluxion has been ſubſtituted in the room of it: but ſtill the ſame general notion is preſerved; for as velocity, at leaſt in its more uſual

usual sense, signifies the degree of quickness with which a body changes its situation in respect of space, so the Fluxion of a quantity signifies the degree of quickness with which the quantity changes its magnitude. Why then is the notion of a Fluxion harder to be understood than that of a velocity? Why may not we make use of the same axioms in relation to one, which all the world allows in relation to the other? Surely some reason ought to be assigned of the difference between the two cases; or it is not fair to make those things peculiar objections against the notion of Fluxions, which equally affect the notion of velocities. And I am apt to think that the reader is mistaken, if he imagines that Sir *Isaac Newton* is peculiarly attacked in the objections of the *Analyst* on this head, and not the common sense of mankind both before and since his time. And what good reason can be assigned why the common notions of all mankind are represented as the peculiar blunders of Sir *Isaac* and his followers, it is not easy to determine. As to the second, *&c.* Fluxions, there may, I allow, seem to be some peculiar difficulty in them to persons that are not used to these subjects; but it seems to me great weakness to imagine that there can be any peculiar objections against them, because tho' the farther we go in the orders of Fluxions, our ideas become more and more complex, yet no new simple ideas are admitted. And when our author asserts, that in order to conceive of a second Fluxion, we must conceive of the velocity of a velocity, and that this is nonsense; he plainly appeals to the sound and not the sense of words. When velocity is considered only as an affection of motion, the velocity of a velocity is nonsense. But then this is not the notion of a second Fluxion; and if by inlarging the idea of velocity, you make it synonymous to the word *Fluxion*, then the velocity of a velocity, however oddly it may sound, is nothing but plain common sense. For the degree of quickness with which any quantity increases, may vary to greater or less with any degree of quickness imaginable; and this last is called the second Fluxion of the quantity. Now if the author of the *Analyst* can shew the absurdity of this notion, he is welcome so to do; but let him not first dress it up in an ambiguous phrase, and then think to confute it by laughing at his own way of understanding that phrase,

PROP. I.

If there be two flowing quantities, one of which flows with a permanent Fluxion, and the other with a continually increasing one, then the Fluxions of these quantities, at any given instant of time, will be in the last ratio of their synchronal changes vanishing at that instant, and in the prime ratio of their synchronal changes which begin to arise at that instant.

1. Let the given instant be the fix'd end of the variable time t, the determinate values of the two flowing quantities at the given instant A and B, their Fluxions then $\dot{y}$ and $\dot{z}$; the flowing quantities themselves y and z; the synchronal changes made in the time t in the quantities y, z, and the Fluxion of z, which continually increases, and I suppose now only as a flowing quantity, whilst the Fluxion of y remains always the same, call respectively y', z', and $\dot{z}'$. Then since $\dot{z}$ is the Fluxion of z at the given instant, or the end of t, and $\dot{z}'$ the change made in it during the time t, and this Fluxion continually increases; therefore $\dot{z} - \dot{z}'$ will always express the Fluxion of z at the beginning of t, whether t be longer or shorter; and consequently, by *Axiom* 4. the ratio of y' to z' will be always

always less than that of $\dot{y}$ to $\dot{z} - \acute{\dot{z}}$, and greater than that of $\dot{y}$ to $\dot{z}$. If therefore $\acute{y} : \acute{z} :: \dot{y} : \dot{z} - x$ always, x will be always less than $\acute{\dot{z}}$. Now as t continually decreases, it is plain, by *Axiom* 1. that the quantities $\acute{y}$, $\acute{z}$, x, $\acute{\dot{z}}$ being always made up of given and flowing quantities, can themselves only vary as flowing quantities, and as such they must vanish, when t vanishes that is at the given instant: [$\acute{y}$ must then vanish, being always equal to the difference between A and y; and A and y are equal at the given instant: for the same reason $\acute{z}$ and $\acute{\dot{z}}$ vanish at the given instant, and x also must vanish, it being all along only a part of $\acute{\dot{z}}$] Wherefore since $\acute{y} : \acute{z} :: \dot{y} : \dot{z} - x$ always, and $\acute{y}$, $\acute{z}$, and x varying only as flowing quantities, vanish at the given instant, (by *Cor.* 3. *Def.* 6.) ult. $\acute{y} : \acute{z} :: \dot{y} : \dot{z}$.

2. AGAIN; let T denote any time that begins at the given instant, $\dot{y}$ and $\dot{z}$ the Fluxions of the quantities at the beginning of the time T; $\acute{y}$, $\acute{z}$, and $\acute{\dot{z}}$, synchronal changes of the quantities and of the Fluxion that increases, made in the time T, and other things being supposed as before. Then by a like reasoning as the foregoing, it will appear, from *Axiom* 5, that if $\acute{y} : \acute{z} :: \dot{y} : \dot{z} + x$, that x is always less than $\acute{\dot{z}}$; and by *Axiom* 1. and *remark* at the end of *Coroll.* 9. that p^{mo} $\acute{y} : \acute{z} :: \dot{y} : \dot{z}$.

3. IF

3. IF the Fluxion of z at the beginning of the time T, receives an instantaneous increment, which is not impossible, nor contrary to any thing supposed in the proposition, then z is to be looked upon as having two Fluxions at the given instant, one of which is that it would have had if no such alteration had been in it; and the other is that it would have had if the same alteration had been made in it before the given instant, and not then. And taking the times t and T so short, that during both there be no other instantaneous alteration made in the Fluxion of z, besides that at the given instant; and following the steps of the foregoing demonstrations, you will find that the ultimate ratio of the changes of y and z vanishing at the given instant, is the same as the ratio of their Fluxions, on the supposition that no alteration had been made in the Fluxion of z at the given instant; and that the prime ratio of the changes of y and z beginning to arise at the given instant, is the same as the ratio of their Fluxions, on the supposition that the alteration in the Fluxion of z had been made before the given instant: all which is manifest, since on the former supposition, the Fluxion of z thro' the time t, and on the latter thro' the time T, varies only as a flowing quantity. Wherefore, &c.

W. W. D.

PROP. II.

IF two quantities flow, one uniformly, and the other with a decreasing Fluxion, or with a Fluxion increasing to a given instant, and then decreasing continually, or *vice versâ*; and even tho' this Fluxion should not always before or after the given instant increase, or always decrease, but for some time only immediately before it, shou'd continue in the same state of increase or decrease, and so likewise immediately after it; yet still the Fluxions of these quantities will be in the last ratio of their synchronal changes

changes vaniſhing at the given inſtant, and in the prime ratio of their ſynchronal changes which begin to ariſe at that inſtant; *i. e.* this is true in whatever manner one of theſe quantities flow. All which is to be made out by following the ſteps of the demonſtration of the foregoing propoſition, having aſſumed your times t and T ſo ſhort, that during either of them there be no alteration of the Fluxion from a ſtate of increaſe to that of decreaſe, or *vice verſâ*.

PROP. III.

If two quantities flow any how, their Fluxions, at a given inſtant of time, will be in the laſt ratio of their ſynchronal changes vaniſhing at that inſtant, and in the firſt ratio of ſynchronal changes then beginning to ariſe.

1. Let the given inſtant be the end of the time t, the flowing quantities z and x, an uniformly flowing quantity y; changes made in theſe quantities in the time t, $\acute{z}$, $\acute{x}$, $\acute{y}$; their Fluxions at the end of t, $\dot{z}$, $\dot{x}$, $\dot{y}$. Then (by Prop. 2.) ult. $\acute{z} : \acute{y} :: \dot{z} : \dot{y}$, and by the ſame ult. $\acute{y} : \acute{x} :: \dot{y} : \dot{x}$; wherefore, by equality, ult. $\acute{z} : \acute{x} :: \dot{z} : \dot{x}$.

2. Let the given inſtant be the beginning of T, the changes made in the time T, $\acute{Z}$, $\acute{X}$, $\acute{Y}$, and their Fluxions at the beginning of T, $\dot{Z}$, $\dot{X}$, $\dot{Y}$.

Then

Then alſo (by Prop. 2.) $p^{mo}\ \acute{Z} : \acute{Y} :: \dot{Z} : \dot{Y}$, and $p^{mo}\ \acute{Y} : \acute{X} :: \dot{Y} : \dot{X}$; and therefore, as before, $p^{mo}\ \acute{Z} : \acute{X} :: \dot{Z} : \dot{X}$. W. W. D.

LEMMA I.

If a determinate ratio be never greater than the greateſt, nor leſs than the leaſt of two variable ratio's, it will not be greater than the greateſt, nor leſs than the leaſt of thoſe ratio's when they are prime or ultimate; *i. e.* if $A : B$ is never greater than $x : y$, nor leſs than $v : z$, it will be a mean alſo between the ultimate or prime ratio of x to y, and v to z; or of the ultimate ratio of x to y, and the prime ratio of v to z, or *vice verſâ*. All which is plain; for, by an evident conſequence from the definitions of prime and ultimate ratio's, the quantities v, z, x, y, may be ſo ſmall, as that their ratio's ſhall differ leſs from their prime and ultimate ratio's than by any aſſignable difference; and therefore if, $x : y$ being the greater of the two ratio's, $A : B$ ſhould be greater than the ultimate ratio of x to y, let the difference between them be D. Then becauſe $x : y$ may be nearer to the ultimate ratio of x to y, than by the difference D, the ratio of $A : B$ may be greater than that of x to y; but it is ſuppoſed to be never greater; which things are inconſiſtent: and therefore $A : B$ is never greater than the ultimate ratio of x to y. And in like manner every thing elſe aſſerted in this *Lemma* may be proved.

Coroll.

Coroll. HENCE if the ultimate ratio of v to z, and the prime ratio of x to y be equal; and the ratio of A to B be never leſs than that of x to y, and never greater than that of v to z, then the ultimate ratio of v to z, and the prime ratio of x to y, will be each equal to that of A to B.

PROP. IV.

SUPPOSING that at the time you ſeek the Fluxions, there be no inſtantaneous change made in them; *i. e.* ſuppoſing the Fluxions you ſeek to be themſelves flowing quantities, or permanent ones, then if you can take ſynchronal increments or decrements of each, made partly before and partly after a given inſtant, in ſuch a manner, that however ſmall they ſhall always be in a given ratio, that ratio will be the ratio of the Fluxions of the quantities at the given inſtant; *i. e.* if the flowing quantities be z and x; and if whilſt z changes from $A-y$ to $A-y+a$, x changes from $B-v$ to $B-v+b$, a being greater than y, and b than v; and the ratio of a to b be a given ratio, then the ratio $a:b$ is the ratio of the Fluxions of z and x at the inſtant that $z=A$ and $x=B$, A and B being permanent quantities: For calling the Fluxions of z and x at that time $\dot{z}$ and $\dot{x}$, ſince v and y are * ſynchronal changes of z and x vaniſhing at the given inſtant, ult. $y:v::\dot{z}:\dot{x}$; and becauſe $a-y$ and $b-v$ are ſynchronal changes of the ſame quantities ariſing at the given inſtant, and there are no inſtantaneous

D changes

* See *Defin.* 5.

changes made in the Fluxions, therefore p^mo^ $a - y : b - v :: \dot{z} : \dot{x}$; but the ratio of a to b, as is evident, is never greater than the greateſt, nor leſs than the leaſt of the ratio's $y : v$, and $a - y : b - v$; and therefore (by *Coroll. Lemm.* preced.) the ratio $a : b$, which is a given ratio, is equal to $\dot{z} : \dot{x}$, which is the ult. ratio of $y : v$, and the prime ratio of $a - y : b - v$. W.W.D.

Remark 1. IT muſt be owned that this propoſition affords only a more indirect way of finding the Fluxions of quantities, and has this diſadvantage above the foregoing, that it not only cannot be applied where there is an inſtantaneous change made in the Fluxions; (it then giving only a mean between the greateſt and leaſt Fluxion of the quantity whoſe Fluxion you ſeek) but alſo when the Fluxion of the quantity you ſeek, compared with a permanent Fluxion, changes from an increaſing to a decreaſing one, or *vice verſâ*. However I thought proper to take notice of it, becauſe in particular caſes it affords the moſt elegant manner of demonſtrating the propoſitions of Fluxions: and the way of demonſtrating from this propoſition, has at leaſt ſome ſimilitude to that which Sir *Iſaac Newton* uſes in *Lemm.* 2. Lib. II. *Princ.*

Remark 2. THE obſervation I made at the end of Prop. 1. That the Fluxion found out from the laſt ratio's and firſt ratio's may poſſibly be different at the ſame inſtant of time, cannot create any difficulty in practice, nor require that we ſhould inveſtigate them both

ways,

ways,* becauſe if the Fluxions of any quantities as found out one way, appear to be flowing quantities, they muſt be ſo in reality, and therefore are always the ſame when found out either way. In what follows, therefore, I ſhall only conſider Fluxions as found from the ultimate ratio's of the vaniſhing changes, but the reader will eaſily ſee that exactly the ſame concluſions would reſult juſt in the ſame manner, by arguing from the prime ratio's of the changes that begin to ariſe at any inſtant.

PROP. V.

THE Fluxion of the ſum of two quantities is, at the end of any given time, or always, equal to the ſum of the Fluxions of each, if they both increaſe or decreaſe together. Let the two flowing quantities be z and y, their ſum, by *Axiom* 1, is a flowing quantity, which call s; then is $z + y = s$ always; and calling the Fluxions of theſe three quantities at the end of any given time $\dot{z}$, $\dot{y}$, $\dot{s}$, and their ſynchronal changes vaniſhing then $\acute{z}$, $\acute{y}$, and $\acute{s}$. Then becauſe z and y increaſe or decreaſe together, $\acute{z} + \acute{y} = \acute{s}$ always; therefore ult. $\acute{s} : \acute{z} + \acute{y} :: 1 : 1$; but (by Prop. 3.) ult. $\acute{z} : \acute{y} :: \dot{z} : \dot{y}$, and therefore

* *i. e.* By conſidering the changes made in the flowing quantities as the differences between their values at the given inſtant and their former values, which differences vaniſh at the given inſtant; and then conſidering them as the differences between their values at the given inſtant, and their ſubſequent values, which differences begin to ariſe at the given inſtant. See *Defin.* 3 and 5.

ult. $\acute{z} + \acute{y} : \acute{y} :: \dot{z} + \dot{y} : \dot{y}$; and therefore from the first and third analogies it follows that ult. $\acute{y} : \acute{s} :: \dot{y} : \dot{z} + \dot{y}$; but (by Prop. 3.) ult. $\acute{y} : \acute{s} :: \dot{y} : \dot{s}$; and therefore because the ultimate ratio of two vanishing quantities is a given ratio $\dot{y} : \dot{z} + \dot{y} :: \dot{y} : \dot{s}$, and $\dot{z} + \dot{y} = \dot{s}$; and consequently the Fluxion of s, the sum of two flowing quantities increasing or decreasing together, is always equal to the sum of the Fluxions of each. W. W. D.

PROP. VI.

IF an equation involve ever so many flowing quantities, turn them all on one side, and then instead of each quantity putting its Fluxion, you will have the equation expressing the relation of the Fluxions when all the quantities flow one way; but if not, change the sign before the Fluxions of those quantities which decrease, and this will give you the equation of the Fluxions.

1. SUPPOSE the quantities to flow all one way, and that the Fluxions of y, z, *&c.* are $\dot{y}$, $\dot{z}$, *&c.* then if $z - y + x = v + r - s$ always, it will follow that $\dot{z} - \dot{y} + \dot{x} - \dot{v} - \dot{r} + \dot{s} = 0$. For from the first equation $z + x + s = v + r + y$ always; and taking $S = z + x$, $\Sigma = S + s$, and $R = v + r$, and $\omega = R + y$ always, 'tis plain that S, Σ, R, ω, which are flowing quantities, by *Axiom* 1. must flow the same way with z, y, *&c.* and therefore (by Prop. 5.)

Prop. 5.) $\dot{S}=\dot{z}+\dot{x}$, $\dot{\Sigma}=\dot{S}+\dot{s}$, $\dot{R}=\dot{v}+\dot{r}$, and $\dot{\omega}=\dot{R}+\dot{y}$; and becauſe $\Sigma=\omega$ always, therefore $\dot{\Sigma}=\dot{\omega}$, and $\dot{R}+\dot{y}=\dot{S}+\dot{s}$; that is, $\dot{v}+\dot{r}+\dot{y}=\dot{z}+\dot{x}+\dot{s}$, or $\dot{z}-\dot{y}+\dot{x}-\dot{v}-\dot{r}+\dot{s}=0$. And the ſame way this part of the propoſition is evident, if there had been ever ſo many flowing quantities in the equation.

2. SUPPOSE ſome of the quantities to decreaſe whilſt others increaſe, and let the equation be $Z-Y+v-x=R+r+T$ always; I ſay then, that if the great letters denote increaſing quantities, and the ſmall ones decreaſing ones, that $\dot{Z}-\dot{Y}-\dot{v}+\dot{X}-\dot{R}-\dot{T}+\dot{r}=0$. For taking $V=2D-v$, and $X=D-x$, and $P=D-r$ always, D being a determinate quantity large enough, it is evident that whilſt v, x, r decreaſe, V, X, P muſt increaſe, and that $Y+R+T+V=Z+X+P$ always. Wherefore by the foregoing part of this propoſition, $\dot{Y}+\dot{R}+\dot{T}+\dot{V}-\dot{Z}-\dot{X}-\dot{P}=0$. But becauſe $V=2D-v$ always, the ſynchronal changes of V and v are always equal, ſince D is a ſtanding quantity; and therefore $\dot{V}=\dot{v}$. And in like manner $\dot{X}=\dot{x}$, and $\dot{P}=\dot{r}$. Putting therefore $\dot{v}$, $\dot{x}$, $\dot{r}$ in the places of $\dot{V}$, $\dot{X}$, $\dot{P}$, it will follow that $\dot{Z}-\dot{Y}-\dot{v}+\dot{x}-\dot{R}-\dot{T}+\dot{r}=0$. And in the ſame way the juſtneſs of the rule

rule might have been demonſtrated (as is evident) had there been ever ſo many more quantities in the equation.

Coroll. THE ſame rule holds, tho' beſides the flowing quantities there be determinate ones in the equation, ſuppoſing their Fluxions $= 0$, as is too evident to need any demonſtration.

N. B. IN practice this change of the ſign is neglected as uſeleſs, but then we conſider the Fluxions of decreaſing quantities as negative, which comes to the ſame thing.

PROP. VII.

LET A be a determinate quantity, z and x flowing ones, their Fluxions $\dot{z}$ and $\dot{x}$; I ſay then, that if $Az = x$ always, $A\dot{z}$ will be $= \dot{x}$ always: for $z : x :: 1 : A$; and therefore the ſynchronal changes, and conſequently the Fluxions of z and x, will be always in the conſtant ratio of 1 to A; *i. e.* $\dot{z} : \dot{x} :: 1 : A$, and conſequently $A\dot{z} = \dot{x}$ always. W.W.D.

PROP. VIII.

LET x and y be two flowing quantities whoſe Fluxions are $\dot{x}$ and $\dot{y}$ always; I ſay then, that if $x^2 = y$ always, that $2x\dot{x} = \dot{y}$ always. Let A be the value of x at a given inſtant, x' and y' ſynchronal changes of x and y vaniſhing at that inſtant, $\dot{X}$ and $\dot{Y}$ their Fluxions at the ſame inſtant;

ſtant; then becauſe $x^2 = y$ always, $2A\acute{x} \mp \acute{x}^2 = \acute{y}$ always, as is evident; and therefore $\acute{x} : \acute{y} :: 1 : 2A \mp \acute{x}$ always, and therefore ult. $\acute{x} : \acute{y} :: 1 : 2A$ (by *Coroll.* 3. *Defin.* 6.) but ult. $\acute{x} : \acute{y} :: \dot{X} : \dot{Y}$ (by Prop. 3.) $\dot{X}$ and $\dot{Y}$ being the Fluxions of x and y at the inſtant their ſynchronal changes $\acute{x}$ and $\acute{y}$ vaniſh; and therefore $\dot{X} : \dot{Y} :: 1 : 2A$, or $2A\dot{X} = \dot{Y}$. But at the given inſtant, when $\dot{x} = \dot{X}$, and $\dot{y} = \dot{Y}$, x is alſo $= A$, and therefore at that inſtant $2x\dot{x} = \dot{y}$. And becauſe the ſame may be proved in the ſame manner at any other inſtant of time, therefore $2x\dot{x} = \dot{y}$ always.

W. W. D.

Or thus:

WHILST x changes from $A - \frac{1}{2}\acute{x}$ to $A - \frac{1}{2}\acute{x} + \acute{x}$, x^2 or y changes from $A^2 - A\acute{x} + \frac{1}{4}\acute{x}^2$ to $A^2 - A\acute{x} + \frac{1}{4}\acute{x}^2 + 2A\acute{x}$; and therefore (by Prop. 4.) the ratio of $\acute{x}$ to $2A\acute{x}$, or 1 to $2A$, being a conſtant ratio, is the ratio of the Fluxions of x to x^2 or y at the inſtant that x becomes $= A$, and $x^2 = A^2$.

PROP.

PROP. IX.

LET x, y, and z be three flowing quantities whose Fluxions are always $\dot{x}$, $\dot{y}$, and $\dot{z}$; I say then, that if $xy = z$ always, and x and y flow the same way, that $\dot{x}y + \dot{y}x = \dot{z}$. But if whilst one of them increases the other decreases, that $\dot{z}$ is equal to $\dot{x}y - \dot{y}x$, or $\dot{y}x - \dot{x}y$, according as z flows the same way with x or y. Let the sum of x and y be v, and its Fluxion $\dot{v}$; then because $x + y = v$ always, $v^2 = x^2 + 2yx + y^2$, *i. e.* $v^2 = x^2 + y^2 + 2z$ always. And if x and y flow the same way, so it is plain must v, v^2, y^2, x^2, and $2z$ do. Taking therefore the Fluxions (by Prop. 6 and 8.) $\dot{x} + \dot{y} = \dot{v}$, and $2v\dot{v} = 2x\dot{x} + 2y\dot{y} + 2\dot{z}$ always; but $2v\dot{v} = \overline{2x + 2y} \times \overline{\dot{x} + \dot{y}} = 2x\dot{x} + 2y\dot{y} + 2\dot{x}y + 2y\dot{x}$ always, and therefore $\dot{z} = \dot{x}y + \dot{y}x$ always.

AGAIN; if z and y flowing one way, x flows the contrary, assume a determinate quantity A large enough, and suppose $A - s = x$ always, the Fluxion of s being $\dot{s}$ always; then will $Ay - sy$ be $= xy$, and consequently $Ay - sy = z$ always. But here it is manifest, because $A - s = x$ always, that s flows the contrary way to x, and therefore the same way with y and z; and therefore

therefore that Ay, sy, and z, and s and y all flow the ſame way, and the contrary way to x. And therefore from the two preceding equations collecting the Fluxions by the foregoing propoſitions, and the part of this that has been proved, $A\dot{y} - \dot{s}y - \dot{y}s = \dot{z}$, and $\dot{s} = \dot{x}$; and inſtead of s and $\dot{s}$ in the former of theſe, putting their values $A - x$ and $\dot{x}$, $A\dot{y} - \dot{x}y - \dot{y}A + \dot{y}x = \dot{y}x - \dot{x}y = \dot{z}$. And laſtly, if z and x flow one way, and y the contrary, in the ſame manner you may prove that $\dot{x}y - \dot{y}x = \dot{z}$.

Coroll. HERE if the Fluxions of decreaſing quantities are conſidered as negative ones, then in all caſes the Fluxion of $xy = \dot{x}y + \dot{y}x$. And from hence the rule in Sir *Iſaac Newton*, from the equation of the Fluents to find the equation of the Fluxions, is eaſily proved.

SECT. III.

I HAVE now proved, I hope, beyond exception, that conſidering Fluxions as the velocities or rates with which quantities increaſe or decreaſe, that their proportions may be always found, if the laſt ratio of their ſynchronal changes vaniſhing at every given inſtant is known.

But I would here obſerve, that whatever falſe metaphyſics there may be in the notion of Fluxions conſidered as velocities, it does not at all affect the general method; as Sir *Iſaac Newton* himſelf has informed us, that inſtead of theſe Fluxions we may make uſe of any other finite quantities found out from the laſt ratio of the ſynchronal increments or decrements of Flowing Quantities; and therefore the Author of the *Analyſt* hardly acted the part of a fair adverſary, in making ſuch a pother about the notion of Fluxions as incomprehenſible, ſince if he had not been able to underſtand them, he might have made uſe of any other quantities he did underſtand in their ſtead. I don't ſay this as if I thought there were any difficulty in the notion of Fluxions thus conſidered; I am on the contrary very poſitive that no man can make any objection againſt a velocity of increaſe or decreaſe in general, that will not as ſtrongly lie againſt a velocity of motion. If quantities may increaſe faſter or ſlower, as well as bodies move faſter or ſlower, there is no greater impropriety in ſaying that one quantity increaſes with a greater velocity than another, than in ſaying that one body moves with a greater velocity than another. But tho' I think there is no difficulty in the notion of Fluxions conſider'd as velocities, yet in order to underſtand equations where Fluxions of different orders are jumbled together, it would be convenient to repreſent all Fluxions not as before, but as quantities of the ſame kind with their Fluents; and therefore I ſhould chuſe to do it, were I to write a treatiſe on this ſubject: And this may be done in the following manner. Take *Defin.* 1, and *Axiom* 2 for the two firſt definitions: Define alſo ultimate

mate ratio as before, and then proceed thus: *Defin.* 4. The Fluxion of an uniformly flowing quantity is the change made in it in a certain given time. *Defin.* 5. The Fluxion of a quantity any how flowing at any given instant, is a quantity found out by taking it to the Fluxion of an uniformly flowing quantity in the ultimate proportion of those synchronal changes which then vanish. And considering Fluxions in this light, all the uses might be made of them as are done under the foregoing notion. But I go on to what the *Analyst* gives me more occasion to take notice of. I might have proved from what went before, that the Fluxion of x is to that of x^n, as 1 to nx^{n-1}. But for the sake of justifying Sir *Isaac Newton*, I shall now take his method.

LET x increase uniformly, and 'tis proposed to find the Fluxion of x^n.

WHILST x by flowing becomes $x+o$, x^n will become $\overline{x+o}|^n$, that is, $x^n+nox^{n-1}+\frac{n^2-n}{2}o^2x^{n-2}+$ &c. and the synchronal augments o and $nox^{n-1}+\frac{n^2-n}{2}o^2x^{n-2}+$ &c. are to one another as $1:nx^{n-1}+\frac{n^2-n}{2}ox^{n-2}$ + &c.

LET now these augments vanish, and their last ratio will be as $1:nx^{n-1}$; and therefore the Fluxion of x is to the Fluxion x^n, as 1 is to nx^{n-1}.

W. W. F.

THIS our author says is no fair and conclusive reasoning, because when we suppose the "increments to vanish, we must suppose their "proportions, their expressions, and every thing "else derived from the supposition of their exi-

 "stence

"ſtence to vaniſh with them." To this I anſwer, that our author himſelf muſt needs know thus much, *viz.* That the leſſer the increment o is taken, the nearer the proportion of the increments of x and x^n will arrive to that of 1 to nx^{n-1}, and that by ſuppoſing the increment o continually to decreaſe, the ratio of theſe ſynchronal increments may be made to approach to it nearer than by any aſſignable difference, and can never come up with it before the time when the increments themſelves vaniſh. And no more nor leſs than this does Sir *Iſaac* mean (as he himſelf informs us) when he ſays, that the laſt ratio of the vaniſhing increments is that of 1 to nx^{n-1}. When therefore our author muſt own that to be true which Sir *Iſaac* intends, what ſignifies it to diſpute whether it be proper to ſpeak of the proportion of the increments as ſtill in being, when the quantities themſelves vaniſh or become $= 0$.

FOR tho', ſtrictly ſpeaking, it ſhould be allowed that there is no laſt proportion of vaniſhing quantities, yet on this account no fair and candid reader wou'd find fault with Sir *Iſaac Newton*, for he has ſo plainly deſcribed the proportion he calls by this name, as ſufficiently to diſtinguiſh it from any other whatſoever: So that the amount of all objections againſt the juſtice of his method in finding out the laſt proportion of vaniſhing quantities, can ariſe to little more than this, that he has no right to call the proportion he finds out according to this method by that name, which ſure muſt be egregious trifling. However, as on this head our author ſeems to talk with more than uſual confidence of the advantage he has over his opponents, and gives

gives us what he says is the * amount of S^r *Isaac's* reasoning in a truly ridiculous light, it will be proper to see on whom the laugh ought to fall, for I am sure somebody must here appear strangely ridiculous. His arguings and illustrations founded on ‖ the *Lemma* he proposes as self-evident, I think I have no occasion to meddle with, because I readily allow whatever consequence he is pleased to draw from it, if it appears that Sir *Isaac*, in order to find the last ratio's proposed, was obliged to make two inconsistent suppositions. To confute which nothing more need be said than barely to relate the suppositions he did make.

1. THEN he supposes that x by increasing becomes $x+o$, and from hence he deduces the relation of the increments of x and x^n.

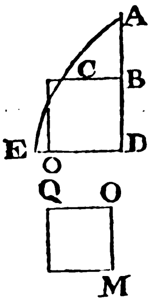
2. AGAIN; in order to find the last ratio of the increments vanishing, he † supposes o to decrease till it vanishes, or becomes equal to nothing. Besides these he makes no other suppositions, and these are evidently no more inconsistent and contradictory, than to suppose a man should first go up stairs, and then come down again. To suppose the increments to be something and nothing at the same time, is contradictory; but to suppose them first to exist, and then to vanish, is perfectly consistent; nor will the consequences drawn from the supposition of their prior existence, if just, be any ways affected by the supposition of their subsequent vanishing, because the truth of the latter supposition no ways

* *Analyst*, p. 21. ‖ Id. p. 20.

† That what is here made the second supposition, is truly the meaning of *Evanescant jam augmenta illa* in Sir *Isaac Newton*, is very plain from the manner in which he supposes quantities to vanish throughout the *Introduction to the Quadratures*.

ways contradicts the truth of the former. To make this more plain, consider what is made out from each supposition: from the first that x has increased by o, this consequence is drawn, that the proportion between the increments of x and x^n, *so long as they exist*, may be expressed by that of 1 to $nx^{n-1} + \frac{n^2-n}{2} o x^{n-2}$, &c. if o always express the increment of x. And this consequence is no ways affected by supposing o continually to decrease, and at length to vanish. But from this last supposition we may gather, that the lesser o is, so much the nearer the ratio of 1 to nx^{n-1} comes to the ratio of the increments; and that by a continual diminution of o, it may come as near to it as you please, but can never equal it before o quite vanishes; and therefore this ratio, and no other whatsoever, agrees to the description which Sir *Isaac* has given of the ultimate ratio of the vanishing increments. His conclusion therefore comes out without the supposition of any thing inconsistent. And our Author, by his way of objecting, seems to make no distinction between two opposite things being done at the same time, and their being done at different ones; between a line's being drawn, and then rubbed out; and lines supposed to be drawn, and not drawn at all; as will appear still farther by considering his objection against the way of finding out the ordinate of a curve from the area and abscisse being perpetually given (which is a problem analogous to that of finding the Fluxion from the Fluent perpetually given, and) which I shall now represent somewhat more distinctly than he has done, and then consider his objection against it.

LET



LET the area $ABC = A$, the abſciſſe $AB = b$, the ordinate $BC = k$, all which are ſtanding quantities as well as the ſquare $QM = M$, and its ſide $QO = q$; and let any other ordinate, with its correſpondent abſciſſe and adjoining area, as DE, AD, AED, be called reſpectively z, x, and V. And the nature of the curve is ſuch, that the area of the curve is to QM as the cube of the adjoining abſciſſe is to the cube of QO; that is, $A \times q^3 = M \times b^3$, and $V \times q^3 = M \times x^3$. From whence I find the area $BCED \times q^3 = M \times \overline{x^3 - b^3}$; and ſuppoſing $BCED =$ to the rectangle Bo, and calling Do, y, I find, inſtead of $BCED$, putting its value, and dividing by $x - b$, that $yq = x^2 + xb + b^2$; which equation I ſee is true, let x be of any magnitude either greater or leſs than b; from whence I conclude it muſt be true alſo when $x = b$. And becauſe I find that y is always a mean between k and z, I conclude that when $x = b$, ſince then $z = k$, that then alſo $y = k$; and therefore that at that time the equation $yq = x^2 + bx + b^2$ muſt degenerate into this, $kq = 3b^2$, which gives me the relation of the ordinate and abſciſſe. W. W. F.

Now in this way of reaſoning, ſays our Author, there is a direct fallacy; becauſe, *firſt*, we are obliged to ſuppoſe that b and x are unequal, without which we could not proceed one ſtep; and, in the *ſecond* place, it is ſuppoſed that they are equal, which is a manifeſt inconſiſtency. But cannot ſo accurate a reaſoner as our Author diſtinguiſh between different times? To ſuppoſe x and b equal and unequal at the ſame time, would

would have been an inconsistency; but to suppose them first unequal, and afterwards to become equal, has not the shadow of difficulty in it. And there is nothing wonderful in it, that from the supposition of both these things happening one after another, I can deduce a conclusion which would not follow from either supposition singly taken; and that the conclusion I have drawn is just, will appear to any one that considers what is distinctly made out in each step of the demonstration.

BY taking x different from b, I find this equation, $yq = x^2 + xb + b^2$, and observe it holds true of whatever magnitude x be taken; the meaning of which may be thus expressed in words at length: There always is a line greater than one and less than the other of the two quantities k and z, so long as they remain different; which multiplied by q is always greater than one, and less than the other of the two quantities $3x^2$ and $3b^2$. Now this proposition is always true whatever becomes of x afterwards, and from it I infer that at the time when $x=b$, and $k=z$, kq must also be equal to $3b^2$. The force of which inference depends upon this plain axiom, That if so long as two quantities remain different, a third be always greater than one and less than the other of them, as soon as by a continued increase or decrease any two of these quantities become equal, all three of them must be equal. The truth of which, were there any manner of occasion for it, might easily be made out by reducing the contrary position to an absurdity.

IN what has been said, I have designedly taken notice only of those objections of the ingenious author of the *Analyst* which relate to the doctrine

doctrine of Fluxions; as for the differential method of *Leibnitz*, I do not undertake its defence, becauſe I think the notion of Fluxions conſidered as velocities, or rather as finite quantities of the ſame kind with their Fluents, as before deſcribed, is much more eaſily conceivable, and frees from difficulty. Nay, I muſt confeſs there ſeems to me to be ſome objection againſt conſidering quantities as generated from moments. What moments, what the *principia jamjam naſcentia finitarum quantitatum*, are in themſelves, I own, I don't underſtand; I can't, I am ſure, eaſily conceive what a quantity is before it comes to be of ſome bigneſs or other; and therefore moments conſidered as parts of the quantities whoſe moments they are, or as really fixed and determinate quantities of any kind, are beyond my comprehenſion, nor do I indeed think that Sir *Iſaac Newton* himſelf did thus conſider them. But when he ſays, for inſtance, that the moment of A^2 is equal to the moment of A into twice A, his meaning is only this, that if the increment of A be continually diminiſhed, the proportion between that into A, and the correſpondent increment of A^2 will approach towards, and at length come nearer to, a ratio of equality than by any aſſignable difference. And thus underſtanding him, the proportion of moments is a phraſe eaſily intelligible, and his meaning very evident in every ſentence where he uſes it, altho', with me, you can't imagine what a moment in itſelf is. 'Tis true indeed, upon this ſuppoſition, it may not be agreeable to the exact accuracy of language, to ſpeak of moments as if they were real intelligible quantities; but I am ſure this is all that can justly be objected against it, and that there is no false logic or metaphysics in it. Whether I have here spoke the sense of Sr *Isaac* himself, or not, I won't be positive, and must leave it to the judgment of the reader conversant in his works; but I think he has given us intimation sufficiently plain, that we may at least thus interpret him if we please, he being not at all solicitous under what notion we consider moments, whilst we take their proportions in that manner he directs us.

OUR Author may perhaps think himself not fairly used, that there has been no more notice taken of second and third &c. Fluxions in this treatise, but it is to no purpose to defend these before the objections against the first are given up; and if the first are well understood, there is no manner of difficulty in the other, except what arises from this consideration, that our ideas become somewhat more complex as we go on to the higher orders of Fluxions. However I shall mention one thing for the sake of young beginners in this science. Suppose $y^2 = x$, and y flows uniformly, then, according to the method of Fluxions, $2y\dot{y} = \dot{x}$, and $2\dot{y}^2 = \ddot{x}$, it may be inquired what is the meaning of this equation taking Fluxion in the notion of velocity, since upon that supposition first and second Fluxions are quantities of different kinds, and therefore have no proportion to one another? To this I answer, that the only meaning of the equation is, that if you take the value

value of $2\dot{y}^2$ and $\ddot{x}$ at two different times, the $2\dot{y}^2$'s are in the same proportion to one another that the $\ddot{x}$'s are. And in like manner are equations always to be underſtood which involve quantities of different ſorts; they don't expreſs the proportion of heterogeneous quantities to one another, but of different values of homogeneous quantities among themſelves. And from hence our Author may, if he pleaſes, receive an anſwer to the latter part of his thirty-firſt query; an odd query, ſurely, to be made by one that pretends to anſwer Sir *Iſaac Newton:* But ſeveral of this kind are to be met with, which ſhew greater prejudice againſt the Mathematicians, than knowledge of the principles which they maintain. And I can't help obſerving, that tho' our Author profeſſes the utmoſt caution as to what he admits as true, and a concern for the utmoſt accuracy in reaſoning, yet neither the one nor the other appears when he is making his objections againſt the Mathematicians. Of this I ſhould not have taken any notice, if his deſign had only been to correct ſome miſtaken notions among the Mathematicians; but as his intention is evidently to run down their method of reaſoning, and to repreſent them as Bigots and Enthuſiaſts in their own ſcience, it is but a piece of juſtice to them to ſhew that their adverſary is as inaccurate in his reaſonings, and as incautious in his aſſertions, and in forming his accuſations, as any of them can be. Of this the reader may have already obſerved ſome inſtances, and I ſhall beg his patience whilſt I mention one or two more.

F 2 1. He

1. HE urges against the Mathematicians with great injustice, I think, that they imagine there may be a proportion between nothings, and yet he himself argues from this principle, *Anal.* p. 18. " Points are undoubtedly equal, as " having no more magnitude one than another, " a limit or point, as such, having no magni- " tude."

2. EVERY reader of the *Analyst* will observe what stress he lays on this maxim, Inaccurate premises cannot infer an accurate conclusion; as if nothing more was necessary than the belief of this principle to confute his adversaries; whereas the inference he would draw from thence, that we therefore cannot, in any case, know the conclusion to be right, if we have argued from the supposition of the truth of what we know to be false, or don't know to be true, is what no rules of logic will justify: because it is possible for me, by virtue of false or inaccurate premises, to gain a conclusion, and at the same time to know that the error in the premises will not cause any error in the conclusion. Thus in Algebra the supposition of a negative quantity, or a quantity less than nothing, is an absurdity, but yet this is no hindrance to the evidence of those conclusions that according to the rules of art are derived from it. And again, to bring an instance more to our present purpose, I'll allow that the supposition of an infinitesimal is absurd, and also that the supposition that an infinitesimal added to a finite quantity does not increase its magnitude is absurd, when an infinitesimal is consider'd as a fix'd determinate quantity;

quantity; and yet, I ſay, in deducing the value of the ſubtangent of a Parabola from theſe ſuppoſitions, I may be ſure that I make a right concluſion; becauſe thoſe quantities which I ſuppoſe to be infinitely ſmall, if they are not ſo will be very ſmall, and thoſe I ſuppoſe to be equal will be very nearly ſo, and therefore the error in the concluſion can be but very ſmall; *i. e.* the ſubtangent will be equal to twice the abſciſſe very nearly. But I am certain that this cannot poſſibly be demonſtrated concerning any finite and determinate quantities, unleſs they are truly and exactly equal, if you don't fix upon ſome determined proportion or difference as the ſtandard of what is nearly equal, but leave every perſon at liberty to chuſe what he will. And from this principle I would obſerve, that the concluſions found by *Leibnitz*'s method of differences may be proved to be true, notwithſtanding the error or inaccuracy of his principles: for ſuppoſing in his method that the mark of equality = does not ſignify truly but nearly equal in the ſenſe before given, and then correct your concluſion by the forementioned axiom, and you may be ſure it is juſt.

3. HE repreſents the Mathematicians as founding their reaſonings on maxims ſhocking to good ſenſe, as particularly when they take it for granted that a finite quantity divided by nothing is infinite, *Query* 16. I ſuppoſe our Author will not pretend to ſay here that he deſigned only to aſk a civil and innocent queſtion, but to charge the Mathematicians with proceeding on ſuch abſurd maxims as this; and if ſo, it is an haſty and ill-grounded charge; 'tis a

charge,

charge, I think, he himſelf can hardly imagine to be true. Does our Author take all Mathematicians to be fools; or did ever any fool imagine that a reaſonable anſwer could be given to this queſtion, How many nothings will fill a quart; and that the proper anſwer is, An infinite number? Yet this is only to ſay that a quart divided by nothing is infinite. 'Tis allowed that this is a rule in Algebra, *that finite divided by nothing is infinite:* but the meaning of it is only this, that if I inquire how big any quantity muſt be taken in order to anſwer any purpoſe, and the anſwer come out according to the rules of Algebra $\frac{1}{0}$ or $\frac{2}{0}$ *&c.* this is a ſign that no finite quantity is large enough. And is there any thing ſhocking in this to the ſenſe of any man, from the porter to the philoſopher? The ſame kind of anſwer ought to be given in relation to other maxims of Algebra, which appear ſtrange to thoſe unaccuſtomed to the phraſes, and really are in themſelves ſenſeleſs expreſſions, as that the ſum or product of two impoſſible quantities may be poſſible. Nay this may be ſaid of the terms multiply and divide, add and ſubtract, as uſed in Algebra. An Algebraiſt never ſcruples to ſubtract a greater quantity from a leſs; but if he really deſigns to do this, he may try till his heart akes before he will be able to accompliſh it, or to know what he is about. All theſe phraſes are therefore to be conſidered merely as terms of art to help the memory in an algebraic proceſs, or ſigns of rules, which from other principles we muſt know to be juſt, and not from any idea we have of the things, which, according to the common

common uſe of the words, an ignqrant man may fancy they are intended to expreſs. 'Tis pity, indeed, theſe things have not been more particularly explained by the writers of Algebra, the want of which may well make it abſtruſe and confounding to beginners. Yet for a perſon to make theſe objections againſt the art or thoſe that underſtand it, is as if I ſhould impute it to the Logicians as a fault in theirs, that they uſe the horrid terms *Barbara*, *Celarent*, &c. Nor do I think the uſe of theſe abſurd expreſſions peculiar to Algebra, but they are frequently uſed where, thro' their being familiar, and the deſign of them eaſily intelligible, they are not thought to have any odd ſound. Thus it is a maxim in Arithmetic that $2 \times 0 = 5 \times 0$, or in general any number multiplied by 0 is $= 0$. Now I ſay that if by this any thing elſe be underſtood than that this is a good rule to go by in managing figures in Arithmetic, I will ſo far venture the laugh of the Public as to declare I don't underſtand it. He that can take two nothings and find their ſum, and then five nothings and find their ſum equal to the former, ſeems to me to be in a fair way to be able to divide by nothing, and find the quotient; and, with a little more pains, may prove that two is equal to five. To multiply by nothing, is as abſurd as to divide by nothing; and to ſuppoſe we can do either, is to imagine nothing to be a real quantity or number: for *nihili nullæ ſunt affectiones.* Here I ſuppoſe our Author agrees with me from his 40th *Query*; but no one will ſay that for this reaſon we ought to reject the maxim conſidered

as

as a rule of Arithmetic, that the product of any number by 0 is = 0.

4thly, *Query* 50. HE represents the disputes and controversies among Mathematicians as disparaging the evidence of their methods: and, *Query* 51. he represents Logics and Metaphysics as proper to open their eyes, and extricate them out of their difficulties. Now were ever two things thus put together? If the disputes of the professors of any science disparage the science itself, Logics and Metaphysics are much more disparaged than Mathematics; why therefore, if I am half blind, must I take for my guide one that can't see at all? And to say the truth, it can hardly be look'd upon as fair to represent Mathematics as disparaged by the disputes of its professors, which insinuates that in this way it is peculiarly disparaged, whereas the quite contrary is true, there being fewer disputes among Mathematicians, as such, than any other persons whatsoever. The disputes of Mathematicians about metaphysical and philosophical principles have nothing to do here; and take away these, hardly any remain.

Lastly, HIS frequent insinuations that Mathematicians don't care to be tried by the rules of good logic, and require indulgence for incorrect and false reasonings, and think that the truth of the premises are proved without more ado by the truth of the conclusion, are entirely groundless. They, as well as all other mortals, desire indulgence for inaccurate expressions, but none for false reasoning; and for following

following the exactest rules of logic, they pride themselves in being the most perfect patterns.

To make a thing plain as a proposition in *Euclid*, is to give it the last degree of evidence. * Nor is it any objection to the justness of their reasoning, that an algebraical note is sometimes to be interpreted, at the end of the process, in a sense which cou'd not have been substituted in the beginning of it; since if quantities themselves are considered as continually changing, the sense of the mark which represents or expresses them must, in order to its doing so, continually change along with them. And they never imagine that any particular supposition can come under a general case which is inconsistent with the reasoning thereof, or any just reasoning whatsoever: tho' they have many times good reason to conclude that particular cases in a general theorem are true, tho' they could not be proved in the same manner with the theorem itself.

To conclude; as I would not be thought, by any thing I have said, to be an enemy to true Logic and sound Metaphysics; and on the contrary think the most general use of the Mathematics is to inure us to a just way of thinking and arguing; it is a proper inquiry, I imagine, for those who have the direction of the education of youth, in what manner mathematical studies may be so pursued as most surely to answer this end: upon which head the hints our Author gives, *Queries* 15, 38, 56, 57, deserve to be consider'd: for

* *Query* 43.

ſo far as Mathematics do not tend to make men more ſober and rational thinkers, wiſer and better men, they are only to be conſidered as an amuſement, which ought not to take us off from ſerious buſineſs.

FINIS.

ERRATA.

PREFACE, pag. 5. l. 11. *for* a perſon *r.* perſons.
26. l. 23. *for* propoſitions *r.* proportions.
32. l. 16. *for* $2y\dot{x}$ r. $2\dot{y}x$.

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