he considers only those terms in the expansion that contain r^m. The first few terms are explicitly given: a modern expansion shows that the term in r^m may be written

$$\sum_{k=0}^{n} \binom{-n+k-1}{k} \binom{n+q-kw-1}{q-kw} r^m,$$

where, following Simpson, we have set $w = 2v + 1$ and $q = m + nv$.

> From which general expression, by expounding m by $0, +1, -1, +2, -2$, &c. successively, the sum of all the chances, whereby the difference of the positive and negative errors can fall within the proposed limits, will be found; which, divided by $r^{-vn} \times (1 - r^w)^n \times (1-r)^{-n}$, will give the true measure of the probability required: from whence the advantage of taking the Mean of several observations might be shewn. [1755, p. 86]

Note that Simpson writes here of 'the true *measure of the probability*', whereas in his *Nature and Laws of Chance*, as we have already noted, he defines *probability* as the ratio of the number of favourable chances to the total number of chances. Similarly, in the introductory *Essai philosophique sur les probabilités* to his *Théorie analytique des probabilités* of 1820 Laplace defines such a ratio as 'la mesure de cette probabilité' in the section *De la probabilité*, and just as 'la probabilité' in the following section[3] (the first of these formulations also being adopted by Poisson in his *Recherches sur la probabilité des jugements en matière criminelle et en matière civile, précédés des règles générales du calcul des probabilités* [1837, p. 31]).

The second proposition differs from the first in having a different progression.

PROPOSITION II.

> Supposing the respective chances, for the different errors which any single observation can admit of, to be expressed by the terms of the series $r^{-v} + 2r^{1-v} + 3r^{2-v} \cdots + \overline{v+1}.r^0 \cdots + 3r^{v-2} + 2r^{v-1} + r^v$ (whereof the coefficients, from the middle one $(v+1)$, decrease, both ways, according to the terms of an arithmetical progression): 'tis proposed to determine the probability, or odds, that the error, by taking the Mean of a given number (t) of observations, exceeds not a given quantity (m/t).
> [1755, p. 87]

Simpson cleverly notes that the given series may be written as the square of the geometric progression

$$r^{-v/2}(1 + r + r^2 + \cdots + r^v),$$

the sum of this progression being easily seen to be given by

$$r^{-v}(1 - r^{v+1})^2/(1 - r)^2.$$

The desired answer is thus obtained as before, with $n = 2t$, $w = v + 1$, and $q = tv + m$.

After particular mention of the case in which $r = 1$ (the problem then being analogous to that of obtaining the points with n dice, each having $2v + 1$ faces — a question solved by Simpson as Problem XXII of his *Nature and Laws of Chance*)[4], Simpson notes that, with respect to the series giving the answer,

> The difference between which and half (w^n), the sum of all the chances, (which difference I shall denote by D), will consequently be the number of the chances whereby the errors in excess (or in defect) can fall within the given limit m: so that $D/\frac{1}{2}w^n$ will be the true measure of the required probability, that the error, by taking the Mean of t observations, exceeds not the quantity m/t, proposed. [1755, p. 90]

(Note that by the 'difference' here is meant $|\text{Series} - (1/2)w^n|$.)

As a specific example, and having noted that the limits expressed by v 'depend on the goodness of the instrument, and the skill of the observer' [1755, p. 91], Simpson supposes that the observations may be relied on to five seconds, and that the chances for the errors $-5''$, $-4''$, ..., $+4''$, $+5''$ are 'respectively proportional to the terms of the series' 1, 2, ..., 2, 1,

> which series seems much better adapted than if all the terms were to be equal, since it is highly reasonable to suppose, that the chances for the different errors decrease, as the errors themselves increase. [1755, p. 91]

Supposing that six observations have been taken, Simpson now finds 'the probability, or chance' of the various errors. He shows, for instance, that the odds are (roughly) $2\frac{2}{3}$ to 1 ($2\frac{1}{2}$ to 1 in the 1757 tract) that the error incurred by taking the mean of six observations exceeds not a single second, these odds being 16 to 20 when only one observation is taken. Proceeding in this way, Simpson concludes the paper by saying

> ... it appears, that the taking of the Mean of a number of observations, greatly diminishes the chances for all the smaller errors, and cuts off almost all possibility of any great ones: which last consideration, alone, seems sufficient to recommend the use of the method, not only to astronomers, but to all others

> concerned in making of experiments of any kind (to which the above reasoning is equally applicable). And the more observations or experiments there are made, the less will the conclusion be liable to err, provided they admit of being repeated under the same circumstances. [1755, pp. 92–93]

In connexion with Simpson's remarks that his advocated method should be used by all experimenters, one might recall Poincaré's writing

> On voit que la méthode des moindres carrés n'est pas légitime dans tous les cas; en général, les physiciens s'en défient plus que les astronomes. Cela tient sans doute à ce que ces derniers, outre les erreurs systématiques qu'ils rencontrent comme les physiciens, ont à lutter avec une cause d'erreur extrêmement importante et qui est tout à fait accidentalle; je veux parler des ondulations atmosphériques. Aussi il est très curieux d'entendre un physicien discuter avec un astronome au sujet d'une méthode d'observation: le physicien, persuadé qu'une bonne mesure vaut mieux que beaucoup de mauvaises, se préoccupe avant tout d'éliminer à force de précautions le dernières erreurs systématiques et l'astronome lui répond: <<Mais vous ne pourrez observer ainsi qu'un petit nombre d'etoiles; les erreurs accidentelles ne disparaitront pas>>. [1903, pp. 241–242]

As we have already said, this paper was essentially incorporated into the 1757 tract, the new material there present being prefaced by the following remarks.

> In the preceding calculations, the different errors to which any observation is supposed subject, are restrained to whole quantities, or a certain, precise, number of seconds; it being impossible, from the most exact instruments, to take off the quantity of an angle to a *geometrical exactness*. But I shall now shew how the chances may be computed, when the error admits of any value whatever, whole or broken, within the proposed limits, or when the result of each observation is supposed to be *accurately* known. [1757a, p. 71]

In the introduction to his 1755 paper, Simpson says that in order to prosecute his design he had

> been obliged to make use of an hypothesis, or to assume a series of numbers, to express the respective chances for the different errors to which any single observation is subject; which series, to me, seems not ill-adapted [1755, p. 83]

and further,