10.3 Commentary on the letter to Canton

Bayes's letter to Canton is undated, but the contents suggest that the conversation referred to, dealing with Simpson's attempt to show the advantage of taking the mean of several observations over the observing of a single value, could have been prompted either by Simpson's *Letter to the Right Honourable* George *Earl of* Macclesfield, *President of the* Royal Society, *on the* Advantage *of taking the Mean of a Number of Observations, in practical Astronomy*, published in the *Philosophical Transactions* in 1756 (in the volume for 1755), or to its republication, with minor changes, as the first part of Simpson's *An Attempt to shew the Advantage arising by Taking the Mean of a Number of Observations, in practical Astronomy*, published in his *Miscellaneous Tracts* of 1757. Bayes's letter does not make the reference clear: if written as a response to the 1755 paper, the lack of any appropriate comment or correction by Simpson in his tract would suggest that he either chose to ignore the letter or was not informed of its contents; if, on the other hand, it was written as a response to the tract, then one cannot expect to find any action being taken.

There are substantial additions to the 1755 paper in the 1757 version, and Clarke is somewhat inaccurate in writing, in her biography of Simpson,

> The fourth paper [in Simpson's *Miscellaneous Tracts*] is an attempt to show mathematically that the mean of a number of astronomical observations is more exact than a single observation. ... In this *Tract* Simpson has somewhat changed the

> original paper, but he gives the same two hypotheses that the distribution of errors might be a rectangle, each magnitude of error being equally probable, or that it might be an equilateral triangle, the probability of an error being inversely proportional to its magnitude. [1929, p. 187]

The 1755 paper consists essentially of two propositions and some examples. We cite the propositions here, with an indication of their proofs, that the reader may have easy access to Simpson's exact results and hence see the point of Bayes's remarks.

PROPOSITION I.

> Supposing that the several chances for the different errors that any single observation can admit of, are expressed by the terms of the progression $r^{-v} \ldots r^{-3}, r^{-2}, r^{-1}, r^{0}, r^{1}, r^{2}, r^{3} \ldots r^{v}$ (where the exponents denote the quantities and qualities of the particular errors, and the terms themselves the respective chances for their happening): 'tis proposed to determine the probability, or odds, that the error, by taking the Mean of a given number (n) of observations, exceeds not a given quantity (m/n).
> [1755, p. 84]

It is perhaps not obvious from the phrase 'to determine the probability, or odds' whether Simpson intends the nouns to be synonymous or whether one is required to determine (a) the probability or (b) the odds (i.e., one of two distinct things). If, however, one turns to Simpson's *Nature and Laws of Chance* of 1740, one finds that *probability* is defined as the ratio of the number of favourable chances to the total number of chances, whereas the term *odds* is introduced in the solution of the following example.

> Imagine a Heap of 16 Counters, whereof 6 are red, and the rest black; and a Person to draw out 2 of them blindfold: To find the Odds that one or both of those shall be red ones.
> [1740, p. 5]

Simpson's solution, viz. odds of 5 to 3 (obtained by his using expectations, and perhaps more expeditiously found by consideration of the complementary event), is also given in terms of the probability 5/8. Thus it seems that *odds* and *probability* are different, rather than synonymous, terms that are connected in the usual way. However in the Note to the solution of Problem XI Simpson writes of 'the Probability, or Odds of winning the Game' [1740, p. 29], and to complicate the matter still further, the solution to this problem begins 'Since the Odds, or Chances, that any assigned Bowl ...'.

One may recall that in his *Essay towards solving a Problem in the Doctrine of Chances* Bayes carefully stated 'By *chance* I mean the same as probability.' Simpson, however, was less precise: he writes, on page 91 (and elsewhere) of his paper of 'the probability, or chance' of various errors, reflecting the title of his 1740 book. And although one might think that *chance* and *chances* have different connotations, one finds in this last work, in a discussion of ordered arrangements of letters, the words 'there is but one Chance, or Way, for all the Letters, or Things, to come out in that Order' [Simpson, 1740, p. 9], and in Problem VII we have

> Supposing a great, but given Number of each of two Sorts of Things to be put promiscuously together; To find how many must be taken out of the Whole, to make it an equal Chance that they shall all come out of one given Sort. [1740, p. 22]

So *chance* is either a *probability* or a *way in which things turn out.*

The ambiguity persists in the use of the plural. Thus in his first definition Simpson has

> The *Probability* of the Happening of an Event is to be understood as the Ratio of the Chances by which that Event may happen, to all the Chances by which it may either happen or fail [1740, p. 1]

whereas in Problem XIII he writes of 'A and B, whose Proportion of Skill, or Chances for winning any assigned Game' (see also Problem XXV), *chances* now being related to ability.

This ambiguity is of course still found today, where one might well say one has three *chances* out of six of getting an even number when a die is tossed, and also assert that the *chance* of an even number is one half.

It would appear then that one should neither seek nor expect to find either precision or uniformity in the use of the terms *chance*, *chances*, *odds*, and *probability* ('variety amidst uniformity', one might be tempted to say in an inversion of Francis Hutcheson's aphorism cited in an earlier chapter); one should perhaps rather be guided by common usage and the presentation of the solution of the problem.

Revenons à ces moutons. To determine the probability sought in Proposition I, Simpson expands $(\sum_{k=-v}^{v} r^k)^n$ by writing it in the form

$$[r^{-v}(1+r+\cdots+r^{2v})]^n \;=\; r^{-nv}(1-r^{2v+1})^n(1-r)^{-n}.$$

Then,

> to find from hence the sum of all the chances whereby the excess of the positive errors above the negative ones can amount, precisely, to a given number m [1755, p. 85]